ME 4555 - Lecture 17 - Inverse Laplace transforms, Pt. 2

Last time, we saw that

$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a} \qquad \text{(first order)}$$

$$\left.\begin{aligned} \mathcal{L}\{e^{-at}\cos bt\} &= \frac{(s+a)}{(s+a)^2+b^2} \\ \mathcal{L}\{e^{-at}\sin bt\} &= \frac{b}{(s+a)^2+b^2} \end{aligned}\right\} \text{(second order)}$$

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and we looked at how to perform PFE when the denominator has <u>real</u> + <u>distinct</u> roots:

$$\frac{B(x)}{A(x)} = \frac{B(x)}{(x-r_1)\cdots(x-r_n)} = \frac{a_1}{x-r_1} + \cdots + \frac{a_n}{x-r_n}$$

then solve for $a_1, \dots, a_n$ so the equality holds.

The a_i's are called the <u>residues</u>. We can actually find them directly. Multiply both sides by $(x-r_i)$ then let $x=r_i$

e.g. if $i=1$,
$$\frac{B(x)}{(x-r_2)\cdots(x-r_n)} = a_1 + a_2\frac{x-r_1}{x-r_2} + \cdots + a_n\frac{x-r_1}{x-r_n}$$

$x=r_1$

$$\boxed{\frac{B(r_1)}{(r_1-r_2)\cdots(r_1-r_n)} = a_1}$$

then repeat for $i=2,3,\dots,n$.

Ex: $$\frac{x^2+1}{x^3+6x^2+11x+6} = \frac{x^2+1}{(x+1)(x+2)(x+3)} = \frac{a}{x+1} + \frac{b}{x+2} + \frac{c}{x+3}$$

For $x=-1$: $$a = \frac{(-1)^2+1}{(-1+2)(-1+3)} = \frac{2}{1\cdot 2} = 1.$$

For $x=-2$: $$b = \frac{(-2)^2+1}{(-2+1)(-2+3)} = \frac{5}{(-1)\cdot(1)} = -5.$$

For $x=-3$: $$c = \frac{(-3)^2+1}{(-3+1)(-3+2)} = \frac{10}{(-2)(-1)} = 5.$$

Therefore: $$\frac{x^2+1}{x^3+6x^2+11x+6} = \frac{1}{x+1} - \frac{5}{x+2} + \frac{5}{x+3}$$

If roots are complex, the same principle holds.

Ex: $$\frac{2x-6}{x^2+2x+5}$$ Roots: $$x = \frac{-2 \pm \sqrt{2^2-4\cdot 5}}{2} = -1 \pm 2i$$

so $$x^2+2x+5 = (x+1+2i)(x+1-2i)$$

For $x=-1+2i$: $$\frac{2(-1+2i)-6}{(-1+2i+1+2i)} = \frac{4i-8}{4i} = 1+2i$$

For $x=-1-2i$: $$\frac{2(-1-2i)-6}{(-1-2i+1-2i)} = \frac{-4i-8}{-4i} = 1-2i$$

So: $$\frac{2x-6}{x^2+2x+5} = \frac{1+2i}{x+1-2i} + \frac{1-2i}{x+1+2i}$$

the "a" and "b" for conjugate roots are always conjugates of one another!

Repeated roots are a special case.

If a pole is repeated k times, e.g. $\frac{1}{(x+1)^k}$,

then the PFE should include: $\left\{ \frac{1}{x+1}, \frac{1}{(x+1)^2}, \dots, \frac{1}{(x+1)^k} \right\}$.

We can obtain the PFE in Matlab using the "residue" command.

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[r, p, k] = residue(num, den)
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r → residues, p → poles, k → quotient polynomial, num → numerator polynomial, den → denominator polynomial.

Ex:

$$p(x) = \frac{x^7 + 10x^6 + 45x^5 + 127x^4 + 231x^3 + 252x^2 + 147x + 19}{x^5 + 7x^4 + 22x^3 + 42x^2 + 41x + 15}$$

Let

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num = [1, 10, 45, 127, 231, 252, 147, 19];
den = [1, 7, 22, 42, 41, 15];
[r,p,k] = residue(num, den);
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Result: $r = \begin{bmatrix} 2 \\ 1+i \\ 1-i \\ 1 \\ -2 \end{bmatrix}$, $p = \begin{bmatrix} -3 \\ -1+2i \\ -1-2i \\ -1 \\ -1 \end{bmatrix}$, $k = [1 \;\; 3 \;\; 2]$

therefore:

$$p(x) = \frac{2}{x+3} + \underbrace{\frac{1+i}{x+1-2i} + \frac{1-i}{x+1+2i}}_{= \frac{2x-2}{x^2+2x+5}} + \frac{1}{x+1} - \frac{2}{(x+1)^2}$$

Final step: apply $\mathcal{L}^{-1}$ to each term.

(1) For real roots, e.g. $\frac{2}{s+3}$, use $\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$.

so $\mathcal{L}^{-1}\left\{\frac{2}{s+3}\right\} = 2e^{-3t}$

$\mathcal{L}^{-1}\left\{\frac{-5}{s-2}\right\} = -5e^{2t}$.

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(2) For complex roots, join the conjugate pairs together and use the formulas:

$$\begin{cases} \mathcal{L}\{e^{-at}\cos bt\} = \dfrac{s+a}{(s+a)^2+b^2} \\ \mathcal{L}\{e^{-at}\sin bt\} = \dfrac{b}{(s+a)^2+b^2} \end{cases}$$

Ex: $$\frac{1+i}{s+1-2i} + \frac{1-i}{s+1+2i} = \frac{2s-2}{s^2+2s+5}$$

$$= \frac{2(s+1)-2\cdot 2}{(s+1)^2+2^2} = 2\left[\frac{(s+1)}{(s+1)^2+2^2}\right] - 2\left[\frac{2}{(s+1)^2+2^2}\right]$$

So $$\mathcal{L}^{-1}\left\{\frac{2s-2}{s^2+2s+5}\right\} = e^{-t}\left(2\cos 2t - 2\sin 2t\right).$$

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(3) For repeated roots, use: $\mathcal{L}\left\{\frac{t^{k-1}}{(k-1)!}e^{-at}\right\} = \frac{1}{(s+a)^k}$

Ex: $$\frac{s-1}{(s+1)^2} = \boxed{\frac{1}{s+1}}\ \boxed{-\ \frac{2}{(s+1)^2}}$$

so $$\mathcal{L}^{-1}\left\{\frac{s-1}{(s+1)^2}\right\} = e^{-t} - 2te^{-t}$$

Ex Find $\mathcal{L}^{-1}\left\{\dfrac{s+10}{s^3-2s^2+10s}\right\}$.

Denominator factors: $s(s^2-2s+10)$.

Does $s^2-2s+10$ have real roots? Use quadratic formula.

$$s = \frac{2 \pm \sqrt{4-4\cdot 10}}{2} = 1 \pm 3i$$

We can directly solve for the final form:

$$\frac{s+10}{s^3-2s^2+10s} = \frac{a}{s} + \frac{bs+c}{s^2-2s+10} = \frac{(a+b)s^2+(c-2a)s+10a}{s^3-2s^2+10s}$$

Equating $s+10 = (a+b)s^2+(c-2a)s+10a$,

we find $a=1$, $b=-1$, $c=3$.

So $$\boxed{\frac{s+10}{s^3-2s^2+10s} = \frac{1}{s} + \frac{-s+3}{s^2-2s+10}} \leftarrow \text{PFE}$$

Now: $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = H(t)$ (or just 1, if we understand that our result is always zero when $t<0$).

and $$\frac{-s+3}{s^2-2s+10} = \frac{-s+3}{(s-1)^2+3^2} = -\left[\frac{s-1}{(s-1)^2+3^2}\right] + \frac{2}{3}\left[\frac{3}{(s-1)^2+3^2}\right]$$

Therefore, $$\boxed{\mathcal{L}^{-1}\left\{\frac{s+10}{s^3-2s^2+10s}\right\} = 1 - e^t\cos(3t) + \frac{2}{3}e^t\sin(3t).}$$